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Computation of Near-Optimal Control Policies by Trajectory Approximation: Hyperbolic-Distributed Parameter Systems with Space-Independent Controls

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Trajectory approximations have been used to compute near-optimal control policies for a number of lumped-parameter systems. In this paper the technique is generalized to distributed-parameter systems. The essence of the method is the approximation of the state and adjoint variables of the problem by a linear combination of spatially dependent trial functions and time-dependent mixing coefficients. Application of the method of weighted residuals to the spatial variations reduces the problem to one involving a system of ordinary differential equations for the mixing coefficients.

Results are presented of an application of the algorithm to the determination of near-optimal control policies for a tubular plug-flow heat exchanger with uniform wall flux forcing. The results compare favorably with the optimal solutions and indicate that the method could be of considerable value in implementing optimal control for a wide class of systems.

The technique of using trajectory approximations to compute near-optimal control policies has been applied to a number of lumped-parameter systems (*8, 10*). The method involves the approximation of the state and adjoint variables of an optimal control problem in terms of linear combinations of trial functions and mixing coefficients. The mixing coefficients are selected to make the trajectory approximations satisfy the system differential equations in an integral average sense by the method of weighted residuals. This results in a programming problem in which the near-optimal control is found by extremizing at each instant of time an objective function subject to functional equality constraints.

The concept of trajectory approximation extends in a natural way to the computation of near-optimal control policies for systems described by partial differential equations. In this case the state and adjoint variables are approximated by linear combinations of spatially dependent trial functions and time-dependent mixing coefficients. Application of the method of weighted residuals to the spatial variations reduces the problem to one involving ordinary differential equations in terms of the mixing coefficients. In the special case where the partial differential equations are linear and the control vector spatially independent, the differential equations for the mixing coefficients form a linear two-point boundary-value problem.

In this paper the trajectory approximation algorithm is developed for distributed-parameter systems with spatially independent controls and applied to the problem of determining near-optimal control policies for a tubular plug-flow heat exchanger with uniform wall flux forcing. The results compare favorably with the solutions for this problem as reported by Koppel, Shih, and Coughanowr (KSC) (*7*), and indicate that the method could be of value in implementing optimal control for a wide class of distributed-parameter systems.

SYSTEM AND NECESSARY CONDITIONS

Necessary conditions for the minimization of an integral performance criterion for control of a class of partial differential equations have been discussed by Koppel et al. (*6, 7*), Chang and Bankoff (*3*), and Denn (*4*). Although it is possible to apply the trajectory approximation algorithm to the general system and performance criterion considered by these authors, the analysis is simplified and the essence of the method preserved by considering the case of a linear hyperbolic equation with a quadratic performance index.

Consider a system of hyperbolic partial differential equations with constant coefficients

$$\frac{\partial x}{\partial t} = Ax + B\frac{\partial x}{\partial r} + Cu(t) \qquad \begin{matrix} t\epsilon[0, t_f] \\ r\epsilon[0, 1] \end{matrix} \tag{1}$$

where $x = x(r, t)$ is an n-component state vector, $u(t)$ is the m-component spatially independent control vector, $m \leq n$, A and B are $n \times n$ matrices, and C is an $n \times m$ matrix.

The conditions at $t = 0$ and $r = 0$ are specified

$$x(r, 0) = x_{ss}(r) \tag{2}$$

$$x(0, t) = x_o(t) \tag{3}$$

The terminal time t_f is fixed and the terminal state $x(r, t_f)$ is free. The quadratic performance index to be minimized is

$$P(u) = \frac{1}{2}\int_0^{t_f}\int_0^1 [x^T(r, t)\,\mu(r, t)\,x(r, t) + u^T\rho(t)\,u(t)]drdt \tag{4}$$

where $\mu(r, t)$ is a positive semidefinite symmetric $n \times n$ matrix and $\rho(t)$ is a positive definite symmetric $m \times m$ matrix.

For this system, the Hamiltonian is defined as

$$H = \frac{1}{2}x^T\mu x + \frac{1}{2}u^T\rho u + p^T\left(Ax + B\frac{\partial x}{\partial r} + Cu\right) \tag{5}$$

where $p = p(r, t)$ is the n-component adjoint vector which satisfies the following equations:

$$\frac{\partial p}{\partial t} = -A^Tp + B^T\frac{\partial p}{\partial r} - \mu x \tag{6}$$

$$p(r, t_f) = 0 \tag{7}$$

$$p^T(1, t)\,B = 0 \tag{8}$$

The optimal control policy $\hat{u}(t)$ must minimize H for all t, $t\epsilon[0, 1]$. It can be shown that this condition yields the following expression for $\hat{u}(t)$ (*7*):

$$\hat{u}(t) = -\rho^{-1}\,C^T\int_0^1 p(r, t)dr \tag{9}$$

The trajectory approximation method developed by Zahradnik and Parkin (*10*) for lumped-parameter systems can be adapted to this system in the following way. The state and adjoint trajectories are approximated in terms of spatially dependent trial functions and time-dependent mixing coefficients

$$x_i(r, t) \approx \overline{x}_i(r, t) = X_{io}(r) + \sum_{j=1}^{N}\alpha_{j+(i-1)N}(t)\,X_{ij}(r) \qquad i = 1, 2, \ldots, n \tag{10}$$

$$p_i(r, t) \approx \overline{p}_i(r, t) = P_{io}(r) + \sum_{j=1}^{N}\alpha_{j+(i-1+n)N}(t)\,P_{ij}(r) \qquad i = 1, 2, \ldots, n \tag{11}$$

where $X_{io}(r)$ and $P_{io}(r)$ satisfy the nonhomogeneous boundary conditions, and $X_{ij}(r)$ and $P_{ij}(r)$ satisfy homogeneous boundary conditions for all values of $\alpha(t)$ the $2nN$-dimensional vector of time-dependent mixing coefficients. The functions $X_{ij}(r)$ and $P_{ij}(r)$ are taken to be the first N members of a complete set of functions for $r\epsilon[0, 1]$.

By requiring that the residuals of the state and adjoint variable equations be orthogonal to the spatially dependent trial functions, a system of integral conditions is generated

$$\int_0^1\left\{\frac{\partial\overline{x}}{\partial t} - A\overline{x} - B\frac{\partial\overline{x}}{\partial r} - C\overline{u}\right\}\begin{Bmatrix} X_{1j} \\ X_{2j} \\ | \\ | \\ X_{nj}\end{Bmatrix}dr = 0 \qquad j = 1, 2, \ldots, N \tag{12}$$

$$\int_0^1\left\{\frac{\partial\overline{p}}{\partial t} + A^T\overline{p} - B^T\frac{\partial\overline{p}}{\partial r} + \mu\overline{x}\right\}\begin{Bmatrix} P_{1j} \\ P_{2j} \\ | \\ | \\ P_{nj}\end{Bmatrix}dr = 0 \qquad j = 1, 2, \ldots, N \tag{13}$$

This procedure is a form of the generalized Galerkin-Kantorovich method (*1, 5*) and results in a set of $2nN$ ordinary differential equations in terms of $\alpha(t)$. The solution of these equations completely specifies $\overline{p}(r, t)$, and

Table 1. Comparison of $\overline{u}(t)$ and $\hat{u}(t)$
$t_f = 0.5, \mu = 2$

$-\overline{u}(t)$

t	$N = 2$	$N = 4$	$N = 6$	$N = 8$	$-\hat{u}(t)$*
0	0.22672	0.27067	0.26669	0.26743	0.26700
0.1	0.15316	0.17938	0.17705	0.17589	0.17644
0.2	0.09506	0.10813	0.10864	0.10798	0.10852
0.3	0.05104	0.05619	0.05860	0.05939	0.05877
0.4	0.01972	0.02107	0.02322	0.02370	0.02359
0.5	0.00000	0.00000	0.00000	0.00000	0.00000

* As reported by Koppel, Shih, and Coughanowr (7).

the near-optimal control policy $\overline{u}(t)$ is obtained by substitution of $\overline{p}(r, t)$ into Equation (9):

$$\overline{u}(t) = -\rho^{-1}(t)\, C^T \int_0^1 \overline{p}(r, t)dr \tag{14}$$

The method is illustrated by computing a near-optimal control policy to drive a tubular heat exchanger from an initial undesired steady state to a new steady state by manipulation of a spatially uniform wall flux.

TUBULAR HEAT EXCHANGER WITH WALL FLUX FORCING

The dynamics of the uniform heat flux exchanger, assuming plug flow in the tube, constant physical properties, and no axial or radial diffusion of heat, can be represented in the following dimensionless form (7):

$$\frac{\partial x}{\partial t} = -\frac{\partial x}{\partial r} + u \tag{15}$$

The state variable $x(r, t)$ represents the deviation of the temperature profile from the final steady state condition, and $u(t)$ represents the deviation of the wall flux from the final flux profile. The inlet temperature $T(0, t)$ is taken to be constant at $T_{ss}(0)$, so that

$$x(0, t) = 0 \tag{16}$$

Initially the exchanger is assumed to be in a steady state corresponding to a steady state control of unity. Thus Equation (15) can be integrated to obtain the initial condition

$$x(r, 0) = r \tag{17}$$

The exchanger is controllable and the optimal policy is chosen to minimize the following quadratic performance index:

$$p(u) = \frac{1}{2} \int_0^{t_f} \int_0^1 (\mu x^2 + u^2) dr dt \tag{18}$$

where μ is a non-negative scalar weight. The Hamiltonian is given by

$$H = \frac{1}{2} (\mu x^2 + u^2) - p \frac{\partial x}{\partial r} + pu \tag{19}$$

The adjoint system equation and its boundary conditions are

$$\frac{\partial p}{\partial t} = -\frac{\partial p}{\partial r} - \mu x \tag{20}$$

$$p(1, t) = 0 \tag{21}$$

$$p(r, t_f) = 0 \tag{22}$$

Application of the stationary condition of Equation (9) yields the optimal control

Table 2. Comparison of $\overline{u}(t)$ and $\hat{u}(t)$,
$t_f = 1, \mu = 2$

$-\overline{u}(t)$

t	$N = 2$	$N = 4$	$N = 6$	$N = 8$	$-\hat{u}(t)$*
0	0.24381	0.29568	0.29148	0.29057	0.29077
0.1	0.17281	0.20421	0.19627	0.19520	0.19535
0.2	0.11713	0.13038	0.12328	0.12510	0.12528
0.3	0.07489	0.07469	0.07309	0.07599	0.07476
0.4	0.04409	0.03624	0.04068	0.04029	0.04007
0.5	0.02272	0.01272	0.01954	0.01647	0.01742
0.6	0.00895	0.00072	0.00571	0.00400	0.00432
0.7	0.00113	−0.00355	−0.00183	−0.00132	0.00197
0.8	−0.00217	−0.00357	−0.00374	−0.00309	0.00336
0.9	−0.00219	−0.00186	−0.00218	−0.00209	0.00210
1.0	0.00000	0.00000	0.00000	0.00000	0.00000

* As reported by Koppel, Shih, and Coughanowr (7).

$$\hat{u}(t) = -\int_0^1 p dr \tag{23}$$

KSC obtained numerical values for the optimal control policy by first solving the state and adjoint equations by the method of characteristics for the uncontrolled system. Then, an improved control policy was obtained from Equation (23). The process was repeated until convergence was obtained, as evidenced by no further decrease in the performance index. The trajectory approximation algorithm will now be applied to this problem to obtain a near-optimal control policy by a noniterative procedure.

The approximations for the state and adjoint variables are formulated as the following power series:

$$\overline{x}(r, t) = \sum_{j=1}^{N} \alpha_j(t) r^j \tag{24}$$

$$\overline{p}(r, t) = \sum_{j=1}^{N} \alpha_{N+j}(t)\, (r-1)^j \tag{25}$$

These series, satisfying boundary conditions (16) and (21), are formed from a complete set of functions and are convenient to use. The initial values of the state mixing coefficients are chosen to satisfy Equation (17)

$$\alpha_1(0) = 1 \tag{26}$$

$$\alpha_j(0) = 0 \quad j = 2, 3, \ldots, N \tag{27}$$

and terminal values of the adjoint mixing coefficients are chosen to satisfy Equation (22)

$$\alpha_j(t_f) = 0 \quad j = N+1,\ N+2, \ldots, 2N \tag{28}$$

The near-optimal control policy for the heat exchanger is obtained by substitution of Equation (25) into Equation (23), and performing the indicated integration

$$\overline{u}(t) = \sum_{j=1}^{N} \frac{(-1)^{j+1}\alpha_{N+j}(t)}{j+1} \tag{29}$$

The orthogonality conditions, Equations (12) and (13), reduce to the following set of $2N$ homogeneous linear ordinary differential equations:

$$\sum_{j=1}^{N} \frac{\dot{\alpha}_j}{i+j+1} = -\sum_{j=1}^{N} \left\{ \frac{j\alpha_j}{i+j} + \frac{(-1)^j \alpha_{N+j}}{(j+1)(i+1)} \right\}$$

$$i = 1, 2, \ldots, N \tag{30}$$

$$\sum_{j=1}^{N} \frac{(-1)^{i+j}\,\dot{\alpha}_{N+j}}{(i+j+1)} = -\sum_{j=1}^{N} \left\{ \frac{\mu(-1)^{i}\, i!\, j!\, \alpha_j}{(i+j+1)!} + \frac{(-1)^{i+j+1}\, j\alpha_{N+j}}{i+j} \right\} \quad i = 1, 2, \ldots, N \qquad (31)$$

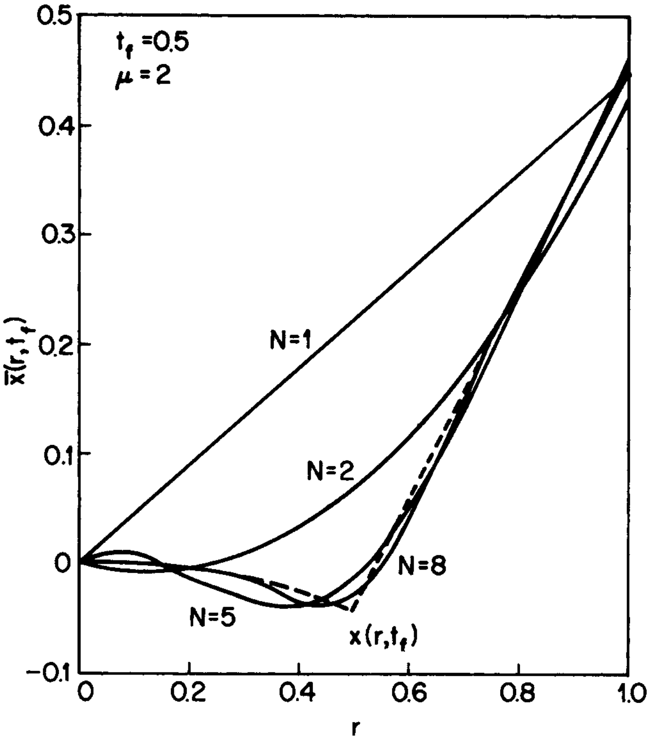

Fig. 1. Convergence of temperature profile.

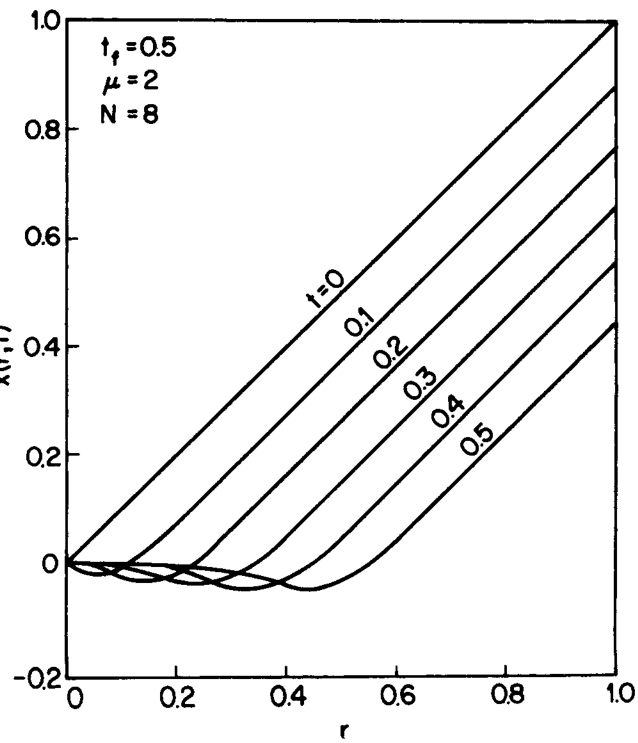

Fig. 2. Temperature profiles.

which can be expressed in terms of the N-component vectors α_S and α_A

$$D^1\, \dot{\alpha}_S = W^1\, \alpha_S + W^2\, \alpha_A \qquad (32)$$

$$D^2\, \dot{\alpha}_A = W^3\, \alpha_S + W^4\, \alpha_A \qquad (33)$$

where

$$\alpha_S = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ | \\ | \\ | \\ \alpha_N \end{bmatrix} \qquad \alpha_A = \begin{bmatrix} \alpha_{N+1} \\ \alpha_{N+2} \\ | \\ | \\ | \\ \alpha_{2N} \end{bmatrix} \qquad (34)$$

and D^1, D^2, W^1, W^2, W^3, W^4 are $N \times N$ matrices whose components are given by

$$d^1_{ij} = \frac{1}{i+j+1} \qquad d^2_{ij} = \frac{(-1)^{i+j}}{i+j+1}$$

$$w^1_{ij} = -\frac{j}{i+j} \qquad w^2_{ij} = \frac{(-1)^{j+1}}{(i+1)(j+1)} \qquad (35)$$

$$w^3_{ij} = \mu\, \frac{(-1)^{i+1}\, i!\, j!}{(i+j+1)!} \qquad w^4_{ij} = \frac{(-1)^{i+j}\, j}{i+j}$$

Equations (32) and (33) can be combined in terms of the $2N$-component vector $\alpha = \begin{pmatrix} \alpha_S \\ \alpha_A \end{pmatrix}$

$$\dot{\alpha} = Z\alpha \qquad (36)$$

where Z is a $2N \times 2N$ matrix given by

$$Z = \left[\begin{array}{c|c} (D^1)^{-1}\, W^1 & (D^1)^{-1}\, W^2 \\ \hline (D^2)^{-1}\, W^3 & (D^2)^{-1}\, W^4 \end{array} \right] \qquad (37)$$

Solution of the boundary-value problem specified by Equation (36) and its boundary conditions, Equations (26) to (28), provides the values of the time-dependent mixing coefficients which are necessary for the determination of the near-optimal control policy for the heat exchanger.

COMPUTATIONAL PROCEDURE

Since Equation (36) is linear, the boundary-value problem can be solved without iteration by using the parameter influence coefficient approach (2, 9). The parameters under consideration are the N missing initial conditions. They are obtained by solving N auxiliary sets of sensitivity equations

$$\dot{\beta}_j = Z\, \beta_j \quad j = 1, 2, \ldots, N \qquad (38)$$

where β_j is a $2N$-component vector and the j^{th} of N vectors defined by

$$\beta_j = \frac{\partial \alpha}{\partial \alpha_{N+j}(0)} \qquad (39)$$

The boundary conditions for Equation (38) are given by

$$\frac{\partial \alpha_i}{\partial \alpha_{N+j}(0)}\,(0) = 1 \qquad \begin{array}{l} i = N + j \\ j = 1, 2, \ldots, N \end{array} \qquad (40)$$

$$\frac{\partial \alpha_i}{\partial \alpha_{N+j}(0)}\,(0) = 0 \qquad \begin{array}{l} i \neq N + j \\ j = 1, 2, \ldots, N \end{array} \qquad (41)$$

Arbitrary values $\alpha_A^*(0)$ are chosen for the missing initial conditions of Equation (36), and the $N + 1$ sets of equations represented by Equations (36) and (38) are solved by a Runge-Kutta-Gill procedure.

An $N \times N$ parameter influence matrix Γ is obtained by evaluation of the parameter influence coefficients at $t = t_f$, and its elements are defined as

$$\gamma_{ij} = \left. \frac{\partial \alpha_{N+i}(t)}{\partial \alpha_{N+j}(0)} \right|_{t=t_f} \tag{42}$$

The missing initial conditions can then be calculated from the relationship

$$\alpha_A(0) = \alpha_A^*(0) + \Gamma^{-1} \epsilon \tag{43}$$

where ϵ is the vector of errors in α_A at the terminal time t_f. Equation (36) is again solved for $\alpha(t)$, this time using the boundary conditions specified by Equations (26), (27), and (43). Since the equations are linear, no further iteration is required. The last N-components of $\alpha(t)$ are then substituted into Equation (29) to obtain the near-optimal control policy.

RESULTS

Near-optimal control policies were calculated on the UNIVAC 1108 in the manner described in the previous section for $t_f = 0.5$, $\mu = 2$ and $t_f = 1$, $\mu = 2$. The control was evaluated at time intervals of $t_f/10$, and a step size of $t_f/50$ was used for the Runge-Kutta-Gill procedure. The approximate state and adjoint variables $\bar{x}(t)$ and $\bar{p}(t)$ were computed at each time interval for increments in r of 0.05.

Tables 1 and 2 present the results obtained for trajectory approximations consisting of up to eight terms. Also reported in Tables 1 and 2 are the results of KSC. The close agreement with the results of KSC for the higher order approximations demonstrates the convergence of the trajectory approximation algorithm. For the case $t_f = 1$, the sign of $\hat{u}(t)$ was apparently incorrectly reported for $t \epsilon [0.7, 1]$, as can be noted by examining plots of $p(r, t)$ versus r in this region. Thus there is control overshoot for this case.

Figure 1 illustrates the behavior of $\bar{x}(t_f)$ as the order of approximation is increased. The dashed curve is the optimal profile obtained by KSC. The state and adjoint variables are plotted in Figures 2 to 5 for $N = 8$. The profiles are similar to those obtained by the method of characteristics, except for the absence of a discontinuity in the spatial derivative at $t = r$ because of the polynomial form of the approximating trajectories. Higher order approximations did not seem warranted since a good degree of convergence had already been obtained. Moreover, computational time increased as the order of approximation increased, going from 1 sec. for $N = 1$ to 11 sec. for $N = 8$.

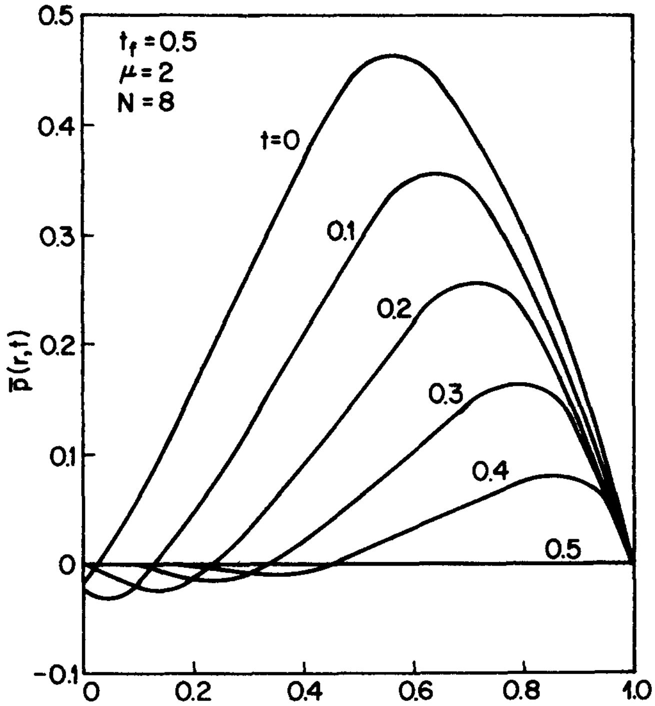

Fig. 3. Adjoint variable profiles.

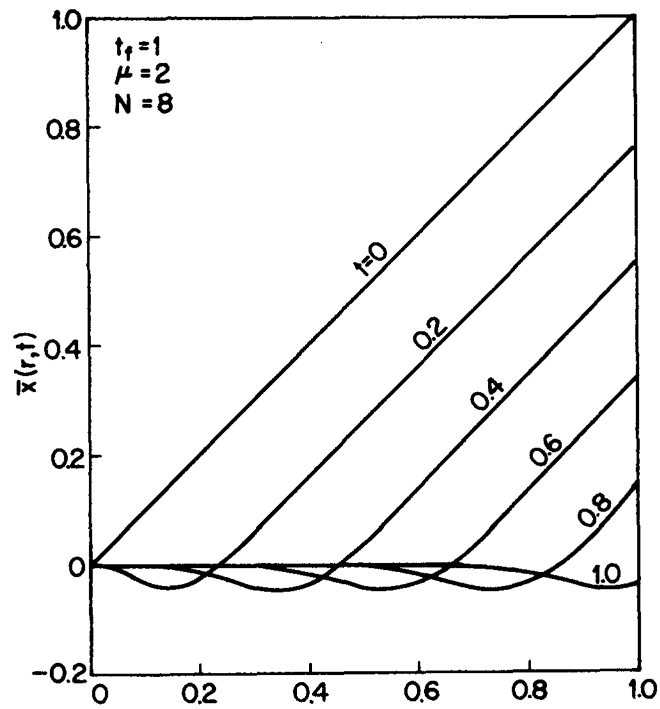

Fig. 4. Temperature profiles.

DISCUSSION

The technique of approximating the trajectories of the state and adjoint variables of an optimal control problem has been shown to be an effective procedure for the computation of near-optimal control policies for a certain class of distributed-parameter systems. By elimination of the spatial variations of the problem through the selection of spatially dependent trial functions, the partial differential equations of the original problem are replaced by systems of ordinary differential equations. This produces two distinct advantages. It obviates the necessity to solve partial differential equations, and it permits the developments of optimal control theory for lumped-parameter systems to be brought to bear on the problem.

The first advantage is of considerable importance, particularly in the case of nonlinear partial differential equa-

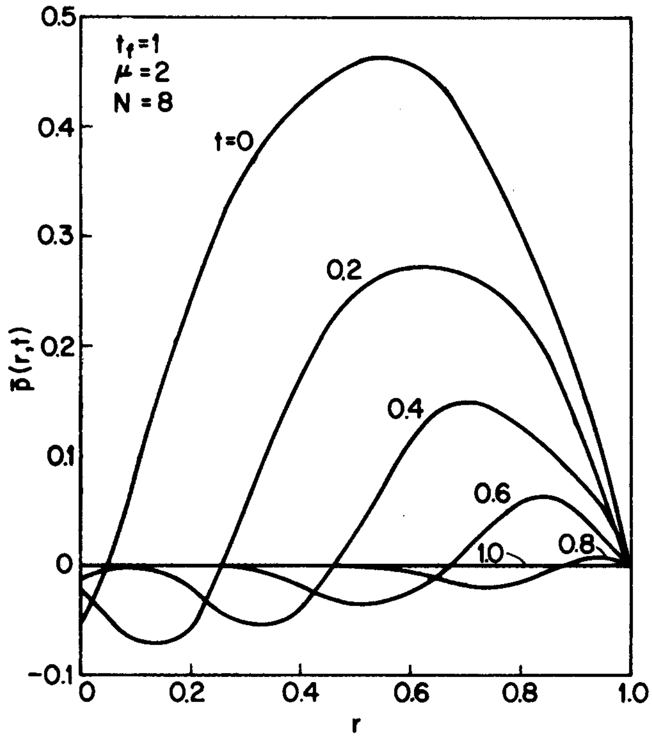

Fig. 5. Adjoint variable profiles.

tions. The separable form of trajectory approximation consisting of time-weighted spatial functions may be applied equally well in the nonlinear case, and herein the method, as in lumped-parameter systems, has its greatest potential. Moreover, if any physical insight as to the nature of the spatial variations is available, it can be incorporated into the trajectory approximations, perhaps reducing the number of terms needed for successful approximation. It is this aspect of the method that distinguishes it from spatial discretization procedures.

The fact that the system of equations, resulting from the application of the method of weighted residuals to the spatial variations of the trajectory approximations, is a system of ordinary differential equations is also of general importance. Optimal control theory for such systems is well established, and many techniques are available to solve the resultant system of equations.

In the specific case of the tubular heat exchanger with wall flux forcing which has been considered in this paper, it can be noted that the method provided reasonably good approximations to the optimal control policy with as few as three or four terms. The state and rather complicated adjoint variable trajectories were likewise well approximated with a reasonable number of terms, in spite of the fact that no special insight was used to select the spatially dependent trial functions.

Finally, the fact that explicit state variable trajectories and control policies are obtained by this method suggests the possibility of obtaining explicit near-optimal feedback control laws for distributed-parameter systems. Koppel et al. (*6, 7*) and Denn (*4*) have determined feedback relationships for a class of distributed-parameter systems with quadratic performance criteria, and it is reasonable to expect that near-optimal analogs of these and other relationships may be developed within the context of the trajectory approximation approach.

NOTATION

A = $n \times n$ constant matrix
B = $n \times n$ constant matrix
C = $n \times m$ constant matrix
D^k = $N \times N$ matrix whose elements d^k_{ij} are defined by Equation (35)
H = Hamiltonian
p = adjoint vector
P = performance index
m = dimension of control vector
n = dimension of state and adjoint vectors
N = order of trajectory approximation
r = spatial coordinate
t = timelike variable
T = fluid temperature
u = control vector
W^k = $N \times N$ matrix whose elements w^k_{ij} are defined by Equation (35)
x = state vector
Z = $2N \times 2N$ matrix as defined by Equation (37)

Greek Letters

α = mixing coefficient vector
β = parameter influence coefficient vector
Γ = parameter influence matrix whose elements γ_{ij} are defined by Equation (42)
ϵ = error vector
μ = weighting function
ρ = weighting function

Superscripts

T = transpose
$\wedge$ = optimal
$-$ = approximate

Subscripts

A = adjoint
f = final time
o = exchanger inlet
ss = steady state
S = state

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